Week 6 - Wednesday

COMP 2230

Last time

- Second order linear homogeneous recurrence relations with constant coefficients
- General recursion
- Started set theory review

Questions?

Assignment 3

Logical warmup

- You have to do an experiment to determine the highest floor on a 100-story building from which a bowling ball can be dropped without breaking.
- You are given two identical bowling balls to carry out your experiment.
- If a ball doesn't break after being dropped, it may be reused without suffering any damage. But if both balls break before you find the highest floor, you're fired.
- What is the least number of times you must drop the bowling balls in order to find the highest floor?

Back to Set Theory

Set equality

- We say that two sets are equal iff they contain exactly the same elements
- Another way of saying this is that $X = Y$ iff $X \subseteq Y$ and $Y \subseteq X$
- Examples:
  - $A = \{n \in \mathbb{Z} \mid n = 2p, p \in \mathbb{Z}\}$
  - $B =$ the set of all even integers
  - $C = \{m \in \mathbb{Z} \mid m = 2q - 2, q \in \mathbb{Z}\}$
  - $D = \{k \in \mathbb{Z} \mid k = 3r + 1, r \in \mathbb{Z}\}$
- Does $A = B$?
- Does $A = C$?
- Does $A = D$?

Set operations

- We usually discuss sets within some superset U called the **universe of discourse**
- Assume that A and B are subsets of U
- The **union** of A and B, written $A \cup B$ is the set of all elements of U that are in either A or B
- The **intersection** of A and B, written $A \cap B$ is the set of all elements of U that are in A and B
- The **difference** of B minus A, written $B - A$ (or sometimes $B \setminus A$), is the set of all elements of U that are in B and not in A
- The **complement** of A, written A^c (or sometimes $\bar{A}$) is the set of all elements of U that are not in A

Examples

- Let $U = \{a, b, c, d, e, f, g\}$
- Let $A = \{a, c, e, g\}$
- Let $B = \{d, e, f, g\}$
- What are:
  - $A \cup B$
  - $A \cap B$
  - $B - A$
  - A^c

The empty set

- There is a set with no elements in it called the **empty set**
- We can write the empty set { } or ∅
- It comes up very often
- For example, $\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$
- The empty set is a subset of every other set (including the empty set)

Disjoint sets and partitions

- Two sets A and B are considered **disjoint** if $A \cap B = \emptyset$
- Sets $A_1, A_2, \ldots A_n$ are **mutually disjoint** (or **nonoverlapping**) if $A_i \cap A_j = \emptyset$ for all $i \neq j$
- A collection of nonempty sets $\{A_1, A_2, \ldots A_n\}$ is a **partition** of set A iff:
  1. $A = A_1 \cup A_2 \cup \ldots \cup A_n$
  2. $A_1, A_2, \ldots A_n$ are mutually disjoint

Power set

- Given a set A, the **power set** of A, written $\wp(A)$ or 2^A is the set of all subsets of A
- Example: $B = \{1, 3, 6\}$
- $\wp(B) = \{\emptyset, \{1\}, \{3\}, \{6\}, \{1,3\}, \{1,6\}, \{3,6\}, \{1,3,6\}\}$
- Let n be the number of elements in A, called the **cardinality** of A
- Then, the cardinality of $\wp(A)$ is 2^n

Cartesian product

- An **ordered n-tuple** $(x_1, x_2, \dots\ x_n)$ is an ordered sequence of n elements, not necessarily from the same set
- The Cartesian product of sets A and B, written $A \times B$ is the set of all ordered 2-tuples of the form (a, b) where $a \in A, b \in B$
- Thus, (x, y) points are elements of the Cartesian product $\mathbb{R} \times \mathbb{R}$ (sometimes written $\mathbb{R}^2$)

Subset Relations

Basic subset relations

- **Inclusion of Intersection:**
  - For all sets A and B
  - $A \cap B \subseteq A$
  - $A \cap B \subseteq B$
- **Inclusion in Union:**
  - For all sets A and B
  - $A \subseteq A \cup B$
  - $B \subseteq A \cup B$
- **Transitive Property of Subsets:**
  - If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Element argument

- The basic way to prove that X is a subset of Y
  1. Suppose that x is a particular but arbitrarily chosen element of X
  2. Show that x is an element of Y

- If every element in X must be in Y, by definition, X is a subset of Y

Procedural versions

- We want to leverage the techniques we've already used in logic and proofs
- The following definitions help with this goal:

1. $x \in X \cup Y \Leftrightarrow x \in X \vee x \in Y$
2. $x \in X \cap Y \Leftrightarrow x \in X \wedge x \in Y$
3. $x \in X - Y \Leftrightarrow x \in X \wedge x \notin Y$
4. $x \in X^c \Leftrightarrow x \notin X$
5. $(x, y) \in X \times Y \Leftrightarrow x \in X \wedge y \in Y$

Example proof

Theorem: For all sets A and B, $A \cap B \subseteq A$

Proof:

- Let x be some element in $A \cap B$
- $x \in A \wedge x \in B$
- $x \in A$
- Thus, all elements in $A \cap B$ are in A
- $A \cap B \subseteq A$ ■

- Premise
- Definition of intersection
- Specialization
- By generalization
- Definition of subset

Laying down the law (again)

Name	Law	Dual
Commutative	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity	$A \cup \emptyset = A$	$A \cap U = A$
Complement	$A \cup A^c = U$	$A \cap A^c = \emptyset$
Double Complement	$(A^c)^c = A$	
Idempotent	$A \cup A = A$	$A \cap A = A$
Universal Bound	$A \cup U = U$	$A \cap \emptyset = \emptyset$
De Morgan's	$(A \cup B)^c = A^c \cap B^c$	$(A \cap B)^c = A^c \cup B^c$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Complements of U and $\emptyset$	$U^c = \emptyset$	$\emptyset^c = U$
Set Difference	$A - B = A \cap B^c$	

Proving set equivalence

- To prove that $X = Y$
  - Prove that $X \subseteq Y$ and
  - Prove that $Y \subseteq X$

Example proof of equivalence

Theorem: For all sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof:

- Let x be some element in $A \cup (B \cap C)$
- $x \in A \vee x \in (B \cap C)$
- **Case 1:** Let $x \in A$
  - $x \in A \vee x \in B$
  - $x \in A \cup B$
  - $x \in A \vee x \in C$
  - $x \in A \cup C$
  - $x \in A \cup B \wedge x \in A \cup C$
  - $x \in (A \cup B) \cap (A \cup C)$
- **Case 2:** Let $x \in B \cap C$
  - $x \in B \wedge x \in C$
  - $x \in B$
  - $x \in A \vee x \in B$
  - $x \in A \cup B$
  - $x \in C$
  - $x \in A \vee x \in C$
  - $x \in A \cup C$
  - $x \in A \cup B \wedge x \in A \cup C$
  - $x \in (A \cup B) \cap (A \cup C)$
- In all possible cases, $x \in (A \cup B) \cap (A \cup C)$, thus $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Proof of equivalence continued

- Let x be some element in $(A \cup B) \cap (A \cup C)$
- $x \in (A \cup B) \wedge x \in (A \cup C)$
- **Case 1:** Let $x \in A$
  - $x \in A \vee x \in B \cap C$
  - $x \in A \cup (B \cap C)$
- **Case 2:** $Let\ x \notin A$
  - $x \in A \cup B$
  - $x \in A \vee x \in B$
  - $x \in B$
  - $x \in A \cup C$
  - $x \in A \vee x \in C$
  - $x \in C$
  - $x \in B \wedge x \in C$
  - $x \in B \cap C$
  - $x \in A \vee x \in B \cap C$
  - $x \in A \cup (B \cap C)$
- In all possible cases, $x \in A \cup (B \cap C)$, thus $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$
- Since both $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ ■

Disproofs and Algebraic Proofs

Disproving a set property

- Like any disproof for a universal statement, you must find a counterexample to disprove a set property
- For set properties, the counterexample must be a specific examples of sets for each set in the claim

Counterexample example

- **Claim:** For all sets A, B, and C, $(A - B) \cup (B - C) = A - C$
- Find a counterexample

Algebraic set identities

- We can use the laws of set identities given before to prove a statement of set theory
- Be extremely careful (even more careful than with propositional logic) to use the law exactly as stated

Algebraic set identity example

Theorem: $A - (A \cap B) = A - B$

Proof:

- $A - (A \cap B) = A \cap (A \cap B)^c$
- $= A \cap (A^c \cup B^c)$
- $= (A \cap A^c) \cup (A \cap B^c)$
- $= \emptyset \cup (A \cap B^c)$
- $= A \cap B^c$
- $= A - B$ ■

Prove or disprove

- For all sets A, B, and C, if $A \nsubseteq B$ and $B \nsubseteq C$, then $A \nsubseteq C$

Prove or disprove

- For all sets A and B, $\left((A^c \cup B^c) - A\right)^c = A$

Russell's Paradox

Naïve set theory

- Set theory is a slippery slope
- We are able to talk about very abstract concepts
  - $\{ x \in \mathbb{Z} \mid x \text{ is prime} \}$
- This is a well-defined set, even though there are an infinite number of primes and we don't know how to find the n^{th} prime number
- Without some careful rules, we can begin to define sets that are not well-defined

Barber Paradox

- Let a barber be the man in Westerville who shaves those men in Westerville who don't shave themselves.
  - Let T be the set of all men in Westerville
  - Let $B(x)$ be "x is a barber"
  - Let $S(x, y)$ be "x shaves y"
  - $\exists b \in T, \forall m \in T, B(b) \wedge (S(b,m) \ \leftrightarrow \ \sim S(m,m))$
- But who shaves the barber?

Russell's Paradox

- Bertrand Russell invented the Barber Paradox to explain to normal people a problem he had found in set theory
- Most sets are not elements of themselves
- So, it seems reasonable to create a set S that is the set of all sets that are not elements of themselves
- More formally, $S = \{ A \mid A \text{ is a set and } A \notin A \}$
- But is S an element of itself?

Escaping the paradox

- How do we make sure that this paradox cannot happen in set theory?
- We can make rules about what sets we allow in or not
- The rule that we use in class is that all sets must be subsets of a defined universe U
- Higher level set theory has a number of different frameworks for defining a useful universe

Upcoming

Next time…

- Halting Problem
- Cardinality
- Relations

Reminders

- Start Assignment 3
  - Due next Friday
- Read 7.4 and 8.1